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Marcin Michalski, Robert Rałowski, Szymon Żeberski
Wrocław University of Science and Technology

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Mycielski theorem

- Let $A \subseteq [0,1] \times [0,1]$ be comeager.
  Then there exists a perfect set P such that
  $$P \times P \subseteq A \cup \Delta.$$
- Assume that $A \subseteq [0,1] \times [0,1]$ has measure 1.
  Then there exists a perfect set P such that
  $$P \times P \subseteq A \cup \Delta.$$

J. Mycielski, Algebraic independence and measure, Fundamenta Mathematicae 61 (1967), 165-169.

A tree $T \subseteq A^{<\omega}$ is called

- a Miller or superperfect tree, if $(\forall\sigma \in T)(\exists\tau \in \omega\text{-split}(T))(\sigma \subseteq \tau)$;
- a Laver tree, if $(\exists\sigma)(\forall\tau \in T)(\tau \subseteq \sigma \vee (\sigma \subseteq \tau \wedge \tau \in \omega\text{-split}(T)))$.

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- a Laver tree, if $(\exists \sigma)(\forall \tau \in T)(\tau \subseteq \sigma \vee (\sigma \subseteq \tau \wedge \tau \in \omega\text{-split}(T)))$.
- uniformly perfect, if for every $n \in \omega$ either $A^n \cap T \subseteq \text{split}(T)$ or $A^n \cap \text{split}(T) = \emptyset$;
- a Silver tree, if $(\forall \sigma, \tau \in T)(|\sigma| = |\tau| \Rightarrow (\forall a \in A)(\sigma^\frown a \in T \Leftrightarrow \tau^\frown a \in T))$.

Mycielski, category case

Laver tree

There exists a dense G_δ set $G \subseteq \omega^\omega$ such that $[T] \not\subseteq G$ for every Laver tree T.

Proof.

$G = \{x \in \omega^\omega : (\exists^\infty n \in \omega)(x(n) = 0)\}$. □

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Miller tree

(Soleki, Spinas) There exists an open dense set $U \subseteq \omega^\omega \times \omega^\omega$ such that $[T] \times [T] \not\subseteq U \cup \Delta$ for every Miller tree T.

S. Solecki, O. Spinas, Dominating and unbounded free sets, Journal of Symbolic Logic 64 (1999), 75-80.

Miller tree..

There exists a dense G_δ set $G \subseteq \omega^\omega \times \omega^\omega$ such that $[T_1] \times [T_2] \nsubseteq G \cup \Delta$ for any Miller trees T_1, T_2.

Proof.

Let $Q = \{q^n : n \in \omega\}$ and $K(q) = \max\{q_1(n), q_2(n) : n \in \omega\}$

$$G = \bigcap_{n\in\omega} \bigcup_{k>n} [q^k \restriction (\mathrm{supp}(q^k) + K(q^k))],$$

□

Silver tree

There exists an open dense set $U \subseteq \omega^\omega \times \omega^\omega$ such that $[T] \times [T] \not\subseteq U \cup \Delta$ for any Silver tree T.

Mycielski, category case, positive result!

For every comeager set G of $\omega^\omega \times \omega^\omega$ there exists a Miller tree $T_M \subseteq \omega^{<\omega}$ and a uniformly perfect tree $T_P \subseteq T_M$ such that

$$[T_P] \times [T_M] \subseteq G \cup \Delta.$$

Mycielski, category case, positive result!

For every comeager set G of $\omega^\omega \times \omega^\omega$ there exists a Miller tree $T_M \subseteq \omega^{<\omega}$ and a uniformly perfect tree $T_P \subseteq T_M$ such that

$$[T_P] \times [T_M] \subseteq G \cup \Delta.$$

but..

There exists a dense G_δ set G such that $[T] \not\subseteq G$ for every uniformly perfect Miller tree T.

Measure case

Miller tree

Let μ be a strictly positive probabilistic measure on ω^ω. Then there exists an F_σ set F of measure 1 such that $[T] \not\subseteq F$ for every Miller tree T.

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uniformly perfect tree

Let F be a subset of $2^\omega \times 2^\omega$ of full measure. Then there exists a uniformly perfect tree $T \subseteq 2^{<\omega}$ satisfying $[T] \times [T] \subseteq F \cup \Delta$.

Small set

$A \subseteq 2^\omega$ is a small set if there is a partition $\mathcal{A}$ of ω into finite sets and a collection $(J_a)_{a\in\mathcal{A}}$ such that $J_a \subseteq 2^a$, $\sum_{a\in\mathcal{A}} \frac{|J_a|}{2^{|a|}} < \infty$ and

$$A = \{x \in 2^\omega : \ (\exists^\infty a \in \mathcal{A})(x \restriction a \in J_a)\}.$$

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Silver tree

There exist a small set $A \subseteq 2^\omega \times 2^\omega$ such that $(A \cap [T] \times [T]) \setminus \Delta \neq \emptyset$ for any Silver tree $T \subseteq 2^{<\omega}$.

Proof.

Let $\{I_n\}_{n\in\omega}$ be a partition of ω, $|I_n| \geq n$.
Clearly, $\{I_n \times I_m\}_{n,m\in\omega}$ forms a partition of $\omega \times \omega$.

$$J_{n,m} = \begin{cases} \emptyset & \text{if } \ n \neq m \\ \{(x,x) : \ x \in 2^{I_n}\} & \text{if } \ n = m \end{cases}$$

$$A = \{(x,y) \in 2^\omega \times 2^\omega : \ (\exists^\infty n \in \omega)(x \restriction I_n = y \restriction I_n)\}$$

$\square$

Silver tree

There exist a small set $A \subseteq 2^\omega \times 2^\omega$ such that $(A \cap [T] \times [T]) \setminus \Delta \neq \emptyset$ for any Silver tree $T \subseteq 2^{<\omega}$.

Silver tree..

Every closed subset of 2^ω of positive Lebesgue measure contains a Silver tree.

Eggleston theorem

Assume that $A \subseteq [0,1] \times [0,1]$ has measure 1.
Then there exists a perfect set P and a perfect set Q, $\lambda(Q) > 0$ such that

$$P \times Q \subseteq A.$$

H. G. Eggleston, Two measure properties of Cartesian product sets, The Quarterly Journal of Mathematics 5 (1954), 108–115.

Eggleston like

ideal $\mathcal{N}$

- If $G \in \mathrm{Bor}([0,1]^2)$ and $G^c \in \mathcal{N}$ then there are $P \in Perf([0,1])$ and $B \in Bor([0,1])$ such that $B^c \in \mathcal{N}$ and $P \times B \subseteq G$.
- If $G \in \mathrm{Bor}([0,1]^2)$ and $G \notin \mathcal{N}$ then there are $P \in Perf([0,1])$ and $B \in Bor([0,1])$ such that $B \notin \mathcal{N}$ and $P \times B \subseteq G$.

Eggleston like

ideal $\mathcal{N}$

- If $G \in \mathrm{Bor}([0,1]^2)$ and $G^c \in \mathcal{N}$ then there are $P \in \mathit{Perf}([0,1])$ and $B \in \mathit{Bor}([0,1])$ such that $B^c \in \mathcal{N}$ and $P \times B \subseteq G$.
- If $G \in \mathrm{Bor}([0,1]^2)$ and $G \notin \mathcal{N}$ then there are $P \in \mathit{Perf}([0,1])$ and $B \in \mathit{Bor}([0,1])$ such that $B \notin \mathcal{N}$ and $P \times B \subseteq G$.

ideal $\mathcal{M}$

- If $G \in \mathrm{Bor}(\mathbb{R}^2)$ and $G^c \in \mathcal{M}$ then there are $P \in \mathit{Perf}(\mathbb{R})$ and $B \in \mathit{Bor}(\mathbb{R})$ such that $B^c \in \mathcal{M}$ and $P \times B \subseteq G$.
- If $G \in \mathrm{Bor}(\mathbb{R}^2)$ and $G \notin \mathcal{M}$ then there are $P \in \mathit{Perf}(\mathbb{R})$ and $B \in \mathit{Bor}(\mathbb{R})$ such that $B \notin \mathcal{M}$ and $P \times B \subseteq G$.

Sz. Żeberski, Nonstandard proofs of Eggleston like theorems, Proceedings of the Ninth Topological Symposium (2001), 353–357.

Fubini product

$A \in \mathcal{I} \otimes \mathcal{J} \Leftrightarrow \{x : A_x \notin \mathcal{J}\} \in \mathcal{I}$

ideal $\mathcal{N} \cap \mathcal{M}$

For every set $G \in Bor(\mathbb{R}^2) \setminus (ctbl \otimes (\mathcal{N} \cap \mathcal{M}))$ there are $P \in Perf(\mathbb{R})$ and $B \in Bor(\mathbb{R}) \setminus (\mathcal{N} \cap \mathcal{M})$ such that $P \times B \subseteq G$.

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ideal $\mathcal{E}$

If $G \subseteq \mathbb{R}^2$ is a Borel subset such that $G^c \in \mathcal{E} \otimes \mathcal{E}$ then there are $P \in Perf(\mathbb{R})$ and $B \in Bor(\mathbb{R})$ such that $B^c \in \mathcal{E}$ and $P \times B \subseteq G$.

ideal $\mathcal{E}$

If $G \subseteq \mathbb{R}^2$ is a Borel subset such that $G^c \in \mathcal{E} \otimes \mathcal{E}$ then there are $P \in \mathit{Perf}(\mathbb{R})$ and $B \in \mathit{Bor}(\mathbb{R})$ such that $B^c \in \mathcal{E}$ and $P \times B \subseteq G$.

Proof.

We start in V. Extend it to $V' \models \mathrm{add}(\mathcal{E}) \geq \omega_3$

Denote $A = \{x : \mathbb{R} \setminus G_x \in \mathcal{E}\}$

$X \subseteq A$, $|X| = \omega_2$. Let $B \subseteq \bigcap_{x \in X} G_x$.

$X \times B \in G$, $\{x : B \subseteq G_x\}$ is Π^1_1 and has cardinality $\geq \omega_2$

$V' \models \exists P \, \exists B \; P \times B \subseteq G$

By Shoenfield absoluteness theorem it is also true in V.

□

Silver tree

For every dense G_δ-set $G \subseteq (2^\omega \times 2^\omega)$ there are a body of a Silver tree $P \subseteq 2^\omega$ and dense G_δ-set $B \subseteq 2^\omega$ such that $P \times B \subseteq G$.

Thank you for your attention!

M. Michalski, R. Rałowski and Sz. Żeberski, Mycielski among trees, Mathematical Logic Quarterly,67 (2021), 271–281